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Computational Methods for Bifurcation Problems with Symmetries on the Manifold*

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COMPUTATIONAL METHODS FOR BIFURCATION PROBLEMS WITH SYMMETRIES ON THE MANIFOLD*

BIN HONG†

Abstract : *This paper is about numerical methods for the determination of bifurcation points of certain steady state multi-parameter problems in the presence of symmetries. A principal tool is the fact that under general conditions the solution set forms a manifold in the space of all state and parameter variables. The reduced manifold with respect to some subsymmetry is introduced. Methods are presented for the local computation of the submanifold of bifurcation points of the same symmetry.*

Key words : *solution manifold, symmetry groups, reduced form, foldpoint calculation*

AMS(MOS) subject classification : *65J15, 58F14*

1. Introduction. Equilibrium problems for many physical systems are modeled by multi–parameter dependent nonlinear equations

$$F(z,\ \lambda) = 0. \tag{1.1}$$

Here z varies in some space Z and characterizes the state of the system while λ denotes the parameter variable allowed to vary in a space Λ of dimension $p \geq 1$. Under fairly general conditions, the regular solution set of (1.1) forms a p–dimensional differentiable manifold and interest centers on determining the foldpoints of the manifold M with respect to the parameter space Λ (see [19]).

Often in physical applications the system (1.1) is covariant with respect to a transformation group G; that is,

$$F(T_z(g)z,\ T_\lambda(g)\lambda)) = S(g)F(z,\ \lambda),\ \text{for all } g \in G, \tag{1.2}$$

where T_z, T_λ and S are group representations of G. When such a symmetry is present, it can aid considerably in the computation of determination of the bifurcation structure, especially at multiple bifurcation points. The computation of a multiple bifurcation point, in general, is difficult and costly. A major advantage of applying group theoretic methods is the possibility of considering the problem in a reduced form reflecting some subsymmetry under which certain multiple bifurcation points reduce to the simple case and hence can be computed by algorithms available for such simple bifurcation points.

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In recent years, a number of papers considered symmetry and symmetry breaking in connection with the computation of bifurcations (see e.g. [5], [12]). But most of them are restricted to one–parameter problems, and even those concerned with multi-parameter problems single out a specific parameter for the computation and hold all other parameters at a fixed value (see e.g. [3]). The aim of this presentation is to use group–theoretic techniques together with a differential–geometric viewpoint to analyse bifurcation phenomena of multi–parameter problems without giving any preference to any one parameter.

Section 2 summarizes the needed background material. We adopt from [19] and [7] the differential–geometric setting for the solution manifolds and then introduce some basic concepts of group theory. This includes some material on the discretization of infinite–dimensional problems under symmetry.

Our approach is based on previous work in the setting of solution manifolds such as methods for following a path on M (see [19]) and for computing simplicial approximations of subsets of M (see [20]), the geometric framework for the numerical study of singular points introduced in [7] and a resulting method for the computation of some type of singular points (see [4]). The key objective of this research is to incorporate group actions into these numerical methods.

Section 3 concerns group actions on solution manifolds. In Section 3, the reduced manifold M_H with respect to some subsymmetry H is introduced and the relation between the original solution manifold M and the reduced solution manifold M_H is discussed. It is shown that the usual trivial group action on the parameter space is not always applicable here and that a more general one has to be introduced. For this, group actions on the parameter space Λ are considered in the case of standard unfoldings in bifurcation theory. For the study of symmetry breaking bifurcations the group action is extended to the so–called cut function and its $(k+1)$–form which in [7] form the basis of the framework for the study of bifurcation phenomena on manifolds. An important question in the study of symmetry breaking is the determination of the choice of the symmetry breaking subgroup. For this, we present a manifold version of the equivariant branching lemma which provides some choices for the symmetry breaking subgroup. In other work on the application of group representation theory to the computation of bifurcations (see e.g. [5], [12]), the symmetry breaking subgroup has to be pre–selected for which the bifurcation point is to be found. We discuss here how to implement numerically the equivariant branching lemma to provide better information for detecting the symmetry breaking subgroup at a bifurcation point. A recent theory about the selection of the choice of the symmetry breaking subgroups developed by Rabier ([17], [18]) is shown to be of considerable help as well. By means of the equivariant branching lemma; that is, by consideration of the original problem in a reduced form reflecting some subsymmetry H, it becomes possible to reduce certain multiple singular points to simple ones and to apply the method for the computation of such simple singular points given in [4]. The set of these singular points with the same symmetry H turns out to form a submanifold of the reduced manifold M_H.

Finally in Section 4, we illustrate the behavior of the methods of Section 3 with an example. In this example comparison of the reduced manifold and of the submanifolds of bifurcation points at a multiple symmetry breaking bifurcation point reflects a rich bifurcation behavior and indicates also some interesting facts about nonconjugate symmetry

breaking bifurcation points.

2. Background. Throughout the paper the following assumption shall hold.

Assumption 2.1 : *(i) X, Y are real Hilbert spaces; (ii) D is an open subset of X; (iii) $F : D \mapsto Y$ is a Fredholm mapping of class $C^r, r \geq 1$ and index $p \geq 1$.*

We consider the equation

$$F(x) = 0, \tag{2.1}$$

and its regular solution set

$$M = \{x : x \in S, F(x) = 0, \text{rge } DF(x) = Y\}, \tag{2.2}$$

which will always be assumed to be nonempty. It is well known that M is a p–dimensional C^r–manifold in X without boundary. For simplicity, the tangent space $T_{x_0}M$ at the point $x_0 \in M$ of this manifold will be identified with the kernel of the derivative $DF(x_0)$ of F; that is, we set

$$T_{x_0}M = \ker DF(x_0) = \{u : u \in X, DF(x_0)u = 0\}, \tag{2.3}$$

whence

$$N_{x_0}M = (T_{x_0}M)^{\perp} = (\ker DF(x_0))^{\perp} = \text{rge } DF(x_0)^* \tag{2.4}$$

may be called the normal space at the point.

A given p–dimensional subspace W of X induces a local coordinate system of M at $x_0 \in M$ if

$$W \cap N_{x_0}M = \{0\}. \tag{2.5}$$

When (2.5) holds, it is always possible to define an isomorphism $A : Y \mapsto W^{\perp}$ of Y onto $W^{\perp}$. Then there exists an open ball $J = B(0, \beta) \subset W$ with radius $\beta > 0$, an open neighborhood $U \subset X$ of x_0, and a unique C^r–function $\eta : J \mapsto Y$ such that $\eta(0) = 0$ and

$$M \cap U = \{x \in X; x = x(t) = x_0 + t + A\eta(t), \forall t \in J\}, \tag{2.6}$$

(see e.g. [19]).

As indicated already by the form of (1.1), many applications involve a natural orthogonal splitting

$$X = Z \oplus \Lambda, \quad Z = \Lambda^{\perp}, \quad \dim \Lambda = p \tag{2.7}$$

of the domain space X into a state subspace Z and a parameter subspace Λ. Then interest often centers on determining the singular points with respect to Λ. Let

$$Z_0 = Z \cap T_{x_0}M, \quad N_0 = \Lambda \cap N_{x_0}M, \tag{2.8}$$

then, a point $x_0 \in M$ is a foldpoint with respect to $X = Z \oplus \Lambda$ if the index $q = \dim Z_0 = \dim N_0$ is nonzero. This integer q is the first singularity index of x_0 (see [7]).

Let $GL(X)$ be the group of all nonsingular linear transformations of X onto itself. A representation of a group G on a vector space X is a homomorphism $T : g \mapsto T(g)$ of G into $GL(X)$, such that

$$T(g_1g_2) = T(g_1)T(g_2), \forall g_1, g_2 \in G. \tag{2.9}$$

In this paper, we consider only finite or compact groups G. Note that for the group representation T of finite or compact groups G on a Hilbert space X, it is always possible to assume T to be unitary, since T is equivalent to a unitary representation (see [21]).

The natural orthogonal splitting (2.7) provides us with an isomorphism $A : Y \mapsto Z$ of Y onto Z. This isomorphism of Y onto Z again denoted by A in turn induces a representation S of G on Y namely

$$S(g) = A^{-1}T(g)A, \forall g \in G. \tag{2.10}$$

This requires that

$$T(g)Z \subset Z, \forall g \in G, \tag{2.11}$$

which is very natural in applications. We shall assume that S is unitary as well.

In line with this we say that the mapping F of Assumption 2.1 is equivariant under the group G or that a covariant group action is defined for the equation (2.1) if the following four conditions hold:

Assumption 2.2 : *(i) G is a finite or compact group; (ii) T is a unitary group representation of G on X such that (2.11) holds; (iii) the representation S of (2.10) induced by T on Y is unitary; (iv) the relation*

$$F(T(g)x) = S(g)F(x), \forall g \in G. \tag{2.12}$$

holds.

As usual, the orbit of a point $x_0 \in X$ under a group G is the set of points

$$O_{x_0} = \{x \in X : x = T(g)x_0, \forall g \in G\}. \tag{2.13}$$

For any point $x \in X$ it is well known that the set

$$G_x = \{g \in G : T(g)x = x\} \tag{2.14}$$

is a subgroup of G. Any subgroup H of G such that $H = G_x$ for some $x \in X$ is called an isotropy subgroup of G. The isotropy subgroups of the points of any orbit are conjugate to each other, that is

$$G_{T(g)x} = gG_xg^{-1}, g \in G, x \in X. \tag{2.15}$$

For any subgroup $H \subset G$, the set

$$X^H = \{x \in X : T(h)x = x, \forall h \in H\}. \tag{2.16}$$

is a subspace of X called the fixed point subspace of H in X. We can analogously define the fixed-point subspace of H in Y. Note that $H_1 \supset H_2$ implies that $X^{H_1} \subset X^{H_2}$. From (2.12) it follows that $F(X^H) \subset Y^H$. This leads to the following important conclusion :

Theorem 2.3 : *Let F_H denote the restriction of F to X^H. Then $x_0 \in X^H$ is a solution of (2.1) if and only if it is a solution of the H-reduced problem*

$$F_H(x) = 0. \tag{2.17}$$

If X is a finite-dimensional space then (2.17) is a lower-dimensional system of equations than (2.1). For the computation this is a most compelling consequence of Theorem 2.3. Some information about the dimension of X^H and the orthogonal projection from X onto X^H is given by the following result.

Theorem 2.4 : *Let G be a finite or compact group acting on X and let $H \subset G$ be a finite subgroup. Then we have*

$$\dim(X^H) = \frac{1}{|H|} \sum_{h \in H} \text{tr } (T(h)). \tag{2.18}$$

Furthermore, the linear operator P_H on X defined by

$$P_H = \frac{1}{|H|} \sum_{h \in H} T(h), \tag{2.19}$$

is the orthogonal projection from X onto X^H.

Theorem 2.4 can be generalized to compact subgroups H. Then the sums in (2.18), (2.19) over the elements of H have to be replaced by Haar integrals over H. For this generalization and the proof of Theorem 2.4 we refer to [11].

In practice, the parameter space is always finite dimensional, but the state-space may be infinite-dimensional. In that case, for the computation, the mapping F in (2.1) needs to be approximated by a finite-dimensional analogue. For this, we follow the presentation in [6] and choose a finite-dimensional subspace Z_h of Z, a discretization projection $P_h \in L(Z)$ such that $\text{rge}P_h = Z_h$. Here h belongs to some index set of small positive scalars. With $X_h = Z_h \oplus \Lambda, Y_h = A^{-1}Z_h \subset Y$, and the projection $Q_h : Y \mapsto Y_h$ specified by $Q_h = A^{-1}P_hA$, the discretized function F_h from the open subset D into Y_h is then defined by

$$F_h : D \mapsto Y_h, F_h(x) = Q_hF(x), x \in D. \tag{2.20}$$

For F_h to preserve the equivariance of F under the group G, we need to know under what condition the group representation S does commute with the projection Q_h.

Theorem 2.5 : ([14, Theorem 3.11]) *If the space X can be decomposed in the form $X = W_1 \oplus W_2$, where W_1 and W_2 are invariant under the representation T of the group G, then the projection operator P on X defined by*

$$Pv = w_1, \text{ for } v = w_1 + w_2, w_i \in W_i, \tag{2.21}$$

satisfies

$$T(g)P = PT(g), \forall g \in G. \tag{2.22}$$

Conversely, if P is a projection operator on X satisfying (2.21) then $X = W_1 \oplus W_2$, where $W_1 =$ rge P and $W_2 = \ker P$ are invariant under T.

By Theorem 2.5, if Z_h is invariant under the group representation T of G on X, then T is interchangeable with P_h. Moreover, Y_h inherits from Z_h the property of being invariant under the induced representation S of G on Y. Thus $S(g)$ commutes with Q_h as well. With the finite dimensional representations $T_h(g) = T(g)|_{X_h}, S_h(g) = S(g)|_{Y_h}$ of G on X_h and Y_h, respectively, we have the "discretized" group action

$$F_h(T_h(g)x) = S_h(g)F_h(x), \forall x \in X_h, \forall g \in G, \tag{2.23}$$

3. Reduced manifolds and symmetry bifurcation. We assume in this section that our problem has been brought into finite dimensional form

$$F(x) = 0, \quad F : \mathbf{R}^n \mapsto \mathbf{R}^m, n = m + p, p \geq 1, \tag{3.1}$$

and consider the derivation of the reduced manifold with respect to a given isotropy subgroup. By the orthogonality of the group representation T we have

$$T^t(g) = T(g)^{-1} = T(g^{-1}), \forall g \in G. \tag{3.2}$$

Moreover, for all $g \in G$ and any $x \in M$ it follows from (2.12) that

$$\begin{aligned} &DF(T(g)x)T(g) = S(g)DF(x), \\ &T(g)^t DF(T(g)x)^t = DF(x)^t S(g)^t, \end{aligned} \tag{3.3}$$

and by (3.2) that

$$T(g)DF(T(g^{-1})x)^t = DF(x)^t S(g). \tag{3.4}$$

From (3.3) we obtain that M is invariant under T; that is, $T(g)(M) \subset M, \forall g \in G$.

As pointed out in Section 2 when F admits a symmetry behavior then it is computationally advantageous to work with the reduced problem with respect to some isotropy subgroup H of G. Let $\dim(X^H) = n_H < \dim X$, $\dim(Y^H) = m_H$, and $F_H : X^H \mapsto Y^H$ the restriction (2.17) of F to X^G. Then we define the corresponding reduced manifold as the regular solution set of F_H; that is,

$$M_H = \{x : x \in X^H, F_H(x) = 0, \text{rge } DF_H(x) = Y^H\}. \tag{3.5}$$

Clearly its dimension is $p_H = n_H - m_H$ and the following theorem provides a connection between M_H and the original manifold M.

Theorem 3.1 : *If $x \in M \cap X^H$ then $x \in M_H$.*

Proof: We need to show that for any $y \in Y^H$ there exists a $v \in X^H$ such that $DF_H(x)v = y$. Because of $x \in M$ we know that for any $y \in Y^H$ there exists a $v_1 \in X$ for which $DF_H(x)v_1 = y$. Since, in addition, $x \in X^H$, it also follows that

$$DF(x)T(h)v_1 = S(h)y, \forall h \in H. \tag{3.6}$$

By applying (3.6) to every group element $h \in H$, and summing the terms we obtain $DF(x)P_H v_1 = Q_H y = y$ and $v = P_H v_1 \in X^H$ which proves the theorem. ▮

The following result is now evident :

Lemma 3.2 : *For each $x \in X^H$, the null spaces* $\ker DF(x)$ *and* $\ker DF(x)^t$ *are invariant under $T(h)$ and $S(h)$, respectively, for each $h \in H$.*

For the study of the solution manifold M it became necessary to introduce a group action on the parameter space Λ. The usual trivial action on the parameter space, used, e.g., in [10], [11], is not always applicable here as the example of the cusp function

$$F(z,\ \lambda,\ \mu) = z^3 - \lambda z + \mu,\ (z,\ \lambda,\ \mu)^T \in \mathbf{R}^3, \tag{3.7}$$

shows which is covariant under the Z_2–symmetry,

$$F(-z,\ \lambda,\ -\mu) = -F(z,\ \lambda,\ \mu). \tag{3.8}$$

It is useful to consider the group action on the parameter space Λ in the case of unfoldings of the type standard in bifurcation theory.

Suppose that $\tilde{F} : \mathbf{R}^m \times \mathbf{R}^1 \mapsto \mathbf{R}^m$ is a C^r–mapping on some open subset D containing a point $(z_0, \mu_0) \in \mathbf{R}^m \times \mathbf{R}^1$ where $\tilde{F}(z_0, \mu_0) = 0$ and $\dim\ker D\tilde{F}(z_0, \mu_0) = r+1 \geq 2$. These conditions imply that $\dim \operatorname{rge} D\tilde{F}(z_0, \mu_0) = m - r \leq m-1$. Let $a_1, \ldots, a_r \in \mathbf{R}^m$ be linearly independent vectors which span $\ker D\tilde{F}(z_0, \mu_0)^t$ and hence which span a complement of $\operatorname{rge} D\tilde{F}(z_0, \mu_0)$ in $\mathbf{R}^m$. Then an unfolding of $\tilde{F}$ is given by

$$F : \mathbf{R}^n = \mathbf{R}^m \times \mathbf{R}^1 \times \mathbf{R}^r \mapsto \mathbf{R}^m, F(z, \mu, \delta) = \tilde{F}(z, \mu) + \delta_1 a_1 + \ldots + \delta_r a_r \tag{3.9}$$

where $\delta = (\delta_1, \ldots, \delta_r)$ and $\Lambda = \mathbf{R}^1 \times \mathbf{R}^r$. By construction we have $\operatorname{rank} DF(x_0, \lambda_0) = m$ and evidently with $n = m + 1 + r, p = r + 1, \lambda = (\mu, \delta), x = (z, \lambda)$, the Assumption 2.1 is satisfied.

Suppose further that the original mapping $\tilde{F}$ is equivariant with respect to a group G; that is, that

$$\tilde{F}(T_z(g)z, \mu) = S(g)\tilde{F}(z, \mu), \forall g \in G, z \in \mathbf{R}^m, \mu \in \mathbf{R}^1, \tag{3.10}$$

where T_z, S are group representations on the spaces Z and Y respectively. Moreover, assume that (z_0, μ_0) belongs to the fixed–point space of G on $\mathbf{R}^m \times \mathbf{R}^1$. We wish to ensure that the unfolded mapping F of (3.9) has the same property; that is, that

$$F(T_z(g)z, T_\lambda(g)\lambda) = S(g)F(z, \lambda), \forall g \in G, z \in \mathbf{R}^m, \lambda \in \mathbf{R}^p, \tag{3.11}$$

where T_λ is a group representation on the space Λ. In order to determine $T_\lambda(g)$ for any $g \in G$, we multiply both sides of (3.9) by $S(g)$,

$$S(g)F(z, \lambda) = S(g)\tilde{F}(z, \mu) + \delta_1 S(g)a_1 + \ldots + \delta_r S(g)a_r. \tag{3.12}$$

By Lemma 3.2 we have $S(g)a_i \in \operatorname{span}(a_1, \ldots, a_r)$, so that

$$S(g)a_i = b_{i1}(g)a_1 + b_{i2}(g)a_2 + \ldots + b_{ir}(g)a_r, i = 1, \ldots, r. \tag{3.13}$$

Then the matrix $B(g) = (b_{ij}(g))$ satisfies $\det B(g) \neq 0$. Hence we can rewrite (3.12) as

$$S(g)F(z,\lambda) = S(g)\tilde{F}(z,\mu) + \delta_1^* a_1 + \ldots + \delta_r^* a_r \tag{3.14}$$

where

$$\begin{pmatrix} \delta_1^* \\ \vdots \\ \delta_r^* \end{pmatrix} = B(g) \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_r \end{pmatrix}. \tag{3.15}$$

By combining (3.11) through (3.15) we see that

$$T_\lambda(g) = \begin{pmatrix} 1 & 0 \\ 0 & B(g) \end{pmatrix}. \tag{3.16}$$

With this T_λ we can define a group representation on X as the direct product of T_z and T_λ; that is

$$T(g)x = (T_z(g)z, T_\lambda(g)\lambda), \forall x \in X, \forall g \in G, \tag{3.17}$$

which then implies that

$$F(T(g)x) = S(g)F(x), \forall x \in X, \forall g \in G. \tag{3.18}$$

The next theorem concerns the relation between the original representation T_Z in (3.10) and the induced representation T_Λ in (3.16).

Theorem 3.3 : *If $F(x) = 0$ and $z = T_z(g)z$ for some $g \in G$, then $\lambda = T_\lambda(g)\lambda$.*

The computation of foldpoints generally involves some augmented equation which is then solved by a Newton–like method. The cutset–idea introduced in [7] provides a natural geometric interpretation of augmentations. In particular, in [7] it was shown that with certain cuts of the solution manifold it is possible to recover many of the results of bifurcation theory required in computational considerations.

In general, we consider cuts of the solution manifold M by the orthogonal complement $W^\perp$ of any given subspace W of X. Thus the cut orthogonal to W through the given point $x_0 \in M$ is the intersection $M \cap (x_0 + W^\perp)$ and therefore consists of the regular points $x \in D$ which satisfy the augmented equations

$$F(x) = 0, \quad \Pi_W(x - x_0) = 0, \tag{3.19}$$

where Π_W is the orthogonal projection of X onto W. If we use the tangential local–coordinate system (2.6) of M at x_0 and choose W as $N_0 = \Lambda \cap N_{x_0}M$ of (2.8), then (3.19) is equivalent with

$$\Pi_{N_0} DF(x_0)^t \eta(t) = 0, t \in J, \tag{3.20}$$

and this suggests the definition of the cut function

$$\begin{aligned} &f : J \subset T_{x_0}M \mapsto N_0 \\ &f(t) = \Pi_{N_0} DF(x_0)^t \eta(t), \ t \in J. \end{aligned} \tag{3.21}$$

The following results address the group equivariance of the cut function.

Lemma 3.4 : *For any point $x_0 \in X^G$, and with $N_0 = \Lambda \cap N_{x_0}M$, we have $T(g)(N_0) \subset N_0, \forall g \in G$.*

The proof is self-evident and need not be detailed here.

Theorem 3.5 : *If $N_0 = \Lambda \cap N_{x_0}M \neq \{\phi\}$ and Π_{N_0} denotes the orthogonal projection onto N_0, then $T(g)\Pi_{N_0} = \Pi_{N_0}T(g), \forall g \in G$.*

Proof : Since $X = Z \oplus \Lambda$, and $X = N_{x_0}M \oplus T_{x_0}M$, it follows that

$$X = Z \oplus (\Lambda \cap T_{x_0}M) \oplus (\Lambda \cap N_{x_0}M). \tag{3.22}$$

Let $W_2 = Z \oplus (\Lambda \cap T_{x_0}M)$, $W_1 = \Lambda \cap N_{x_0}M = N_0$. Then Lemma 3.4 implies that W_1 is invariant under T and because of $Z = \Lambda^{\perp}, N_{x_0}M = (T_{x_0}M)^{\perp}$ we have $W_2 = W_1^{\perp}$. Hence, W_2 is also invariant under T and (2.22) of Theorem 2.5 applies to $P = \Pi_{N_0}$. ∎

Theorem 3.6 : *Let $x_0 \in M \cap X^G$. The cut function f defined by (3.21) at x_0 satisfies*

$$f(T(g)t) = T(g)f(t), \forall g \in G, \forall t \in J. \tag{3.23}$$

Proof : (a) We show first that the function η in the characterization (2.6) of a local coordinate system satisfies

$$\eta(T(g)t) = S(g)\eta(t), \forall g \in G, \forall t \in J. \tag{3.24}$$

In the proof of Theorem 2.2 ([19, Theorem 4.4]) the inverse function theorem ensures the existence of an open ball $J = B(0, \beta)$ with radius $\beta > 0$, of a neighborhood $V \subset Y$ of the origin, and of a unique C^r–function $\eta : J \mapsto V$ such that

$$H(t, \eta(t)) = F(x_0 + t + DF(x_0)^t\eta(t)) = 0, \forall t \in J. \tag{3.25}$$

Since T is unitary, we have $T(g)J \subset J$, and

$$\begin{aligned} S(g)H(t, \eta(t)) &= S(g)F(x_0 + t + DF(x_0)^t\eta(t)) \\ &= F(x_0 + T(g)t + DF(x_0)^t S(g)\eta(t)) \\ &= H(T(g)t, S(g)\eta(t)) = 0. \end{aligned} \tag{3.26}$$

On the other hand, by replacing $t \in J$ by $T(g)t \in J$ in (3.25) it follows that

$$H(T(g)t, \eta(T(g)t)) = 0. \tag{3.27}$$

Since $T(g)t \in J$, the solution of (3.25) for $T(g)t$ is unique and hence

$$\eta(T(g)t) = S(g)\eta(t).$$

(b) From (a) and Theorem 3.5, we obtain that

$$\begin{aligned} T(g)f(t) &= T(g)\Pi_{N_0}DF(x_0)^t\eta(t) \\ &= \Pi_{N_0}DF(x_0)^t S(g)\eta(t) \\ &= \Pi_{N_0}DF(x_0)^t\eta(T(g)t) = f(T(g)t), \end{aligned} \tag{3.28}$$

which proves the theorem. ∎

It was shown in [7] that the cut function f always satisfies

$$f(0) = 0, Df(0) = 0. \tag{3.29}$$

Accordingly, the second singularity index of the foldpoint $x_0 \in M$ is defined (see [7]) as the number $k \geq 1$ such that

$$D^j f(0) = 0, j = 0, 1, \ldots, k, D^{k+1} f(0) \neq 0, \tag{3.30}$$

if there exists such a k, else $k = \infty$. With this, the triple (p, q, k) consisting of the Fredholm index p and the two singularity indices q and k forms a classification for the foldpoints of M.

For the determination of the form of the cuts the generalized Morse Theorem due to Buchner, Marsden and Schecter [1] is applied in [7]. There it was shown that the form of the cut orthogonal to N_0 of (2.8) at the foldpoint x_0 of type (p, q, k) is essentially determined by the zeros of the $(k+1)$–form

$$\begin{aligned} &Q : J \subset T_{x_0} M \mapsto N_0, \\ &Q(t) = \frac{1}{(k+1)!} D^{k+1} f(0)(t, t, \ldots, t), t \in J. \end{aligned} \tag{3.31}$$

More specifically, if Q is regular on its zero set; in other words, if $DQ(t)$ is surjective for each nonzero $t \in Q^{-1}(0)$, then locally near x_0 the cutset is C^1–diffeomorphic to the zero set of Q. In line with this, a foldpoint of M of type (p, q, k) will be called nondegenerate if its associated $(k+1)$–form (3.31) is regular on its zero set.

Theorem 3.7 : *The $(k+1)$–form Q associated with the cut function f in Theorem 3.6 is equivariant under G,*

$$Q(T(g)t) = T(g)Q(t), \forall g \in G, \forall t \in J. \tag{3.32}$$

Proof : By Taylor's theorem we can write

$$f(t) = Q(t) + o(||x||^{k+1}). \tag{3.33}$$

and for $t \neq 0$, we have

$$\begin{aligned} f(T(g)t) &= Q(T(g)t) + o(||T(g)t||^{k+1}) \\ &= Q(T(g)t) + o(||t||^{k+1}), \end{aligned} \tag{3.34}$$

$$T(g)f(t) = T(g)Q(t) + o(||t||^{k+1}). \tag{3.35}$$

Now divide both sides of (3.34) and (3.35) by $||t||^{k+1}$ and set $v = \frac{t}{||t||}$. Then for $t \to 0$ it follows, in view of (3.23), that

$$Q(T(g)v) = T(g)Q(v), \forall g \in G, v \in T_{x_0} M, ||v|| = 1, \tag{3.36}$$

and by homogeneity the condition $||v|| = 1$ can be dropped in (3.36).∎

Remark : The linearly independent vectors $v \in Q^{-1}(0), v \neq 0$ are called bifurcation directions. By Theorem 3.7, if v is the zero of Q, so is the $T(g)v, g \in G$, that means, the representation T acts on $Q^{-1}(0)$ by permuting the bifurcation directions under symmetry. Therefore, linearly independent bifurcation directions $v \in Q^{-1}(0), ||v|| = 1$ correspond to different orbits through x_0 on M.

If x_0 is a foldpoint in $X^G \cap M$ it may occur that the bifurcation directions at x_0 belong to X^H for some isotropy subgroup H of G. In that case the symmetry group of equation (2.1) remains unchanged but the bifurcation directions spontaneously break symmetry (in the absence of an external symmetry breaking perturbation).

An important question in the study of symmetry breaking is the determination of the subgroup H of G which breaks the G-symmetry. For this, we present a manifold version of the equivariant branching lemma which provides some choices for the symmetry breaking subgroup.

Theorem 3.8 : *Suppose that $x_0 \in M \cap X^G$ is a foldpoint of type $(p, q, 1)$, and H is an isotropy subgroup of G. Let $Z_0 = Z \cap T_{x_0}M$. If $\dim(Z_0 \cap X^H) = 1$, say, $Z_0 \cap X^H = \mathrm{span}(v_0)$, then v_0 is a bifurcation direction in M.*

Proof : It was shown in [4] that if x_0 is a foldpoint of type $(p, 1, 1)$; that is, if $\dim Z_0 = 1$, and $Z_0 = \mathrm{span}(u_0)$, then u_0 is the bifurcation direction. The key point here is to reduce the original multiple foldpoint problem of type $(p, q, 1)$ to the H-reduced simple foldpoint problem of type $(p_H, 1, 1)$. If we can show that $Q_H(v_0) = 0$ then by the equivariance of Q (Theorem 3.7), we also have $Q(v_0) = 0$. More specifically, we need to show that

$$q_H = \dim\,(T_{x_0}M_H \cap Z^H) = 1, \tag{3.37}$$

where

$$Z^H = \{z \in Z; T(h)z = z, \forall h \in H\} \tag{3.38}$$

is the fixed-point subspace of H on Z. Since, by definition, $F_H(x) = F(P_H x)$, it follows that

$$DF_H(x) = DF(P_H x)P_H = DF(x)P_H, \forall x \in X^H, \tag{3.39}$$

which proves the identity

$$Z^H \cap \ker DF_H(x) = X^H \cap (Z \cap \ker DF(x)) = X^H \cap Z_0. \tag{3.40}$$

This, together with the assumptions $x_0 \in M \cap X^G \subset M \cap X^H$ and $\dim(X^H \cap Z_0) = 1$, implies that (3.37) holds. ∎

Remark : It can be shown that if Q is regular at v_0, then the bifurcated branch tangent to v_0 preserves the symmetry of the isotropy subgroup of v_0 (see [16]).

For the numerical implementation of Theorem 3.8 note that

$$Z_0 = Z \cap T_{x_0}M = \ker D_z F(x_0), \tag{3.41}$$

and

$$X^H \cap Z_0 = Z_0^H = \{z \in Z_0; T(h)|_{Z_0} z = z, \forall h \in H\}. \tag{3.42}$$

If $(P_0)_H$ denotes the orthogonal projection from Z_0 onto Z_0^H then by (2.18) and (2.19) we have

$$\begin{aligned} \dim(Z_0^H) &= \frac{1}{|H|} \sum_{h \in H} \text{tr } (T(h)|_{Z_0}) \\ &= \text{tr } (\frac{1}{|H|} \sum_{h \in H} T(h)|_{Z_0}) = \text{tr}((P_0)_H). \end{aligned} \tag{3.43}$$

Lemma 3.9 : (see Chapter I,§2 of [15]) *Let A be a linear operator on a finite–dimensional space U and V a subspace of U that is invariant under A. Moreover, let P be a projection of U onto V and A_V the restriction of A to V. Then we have*

$$\text{tr}(A_V) = \text{tr}(AP). \tag{3.44}$$

Let $(P_Z)_H$ be the orthogonal projection from Z onto Z_H and $\Pi_{\ker D_z F(x_0)}$ the orthogonal projection from Z onto $\ker D_z F(x_0)$. Then, by Lemma 3.9, (3.43) becomes

$$\dim(X^H \cap Z_0) = \text{tr}((P_Z)_H \Pi_{\ker D_z F(x_0)}), \tag{3.45}$$

where

$$(P_Z)_H = \frac{1}{|H|} \sum_{h \in H} T_z(h) \tag{3.46}$$

and $\Pi_{\ker D_z F(x_0)}$ is available from the singular value decomposition (SVD) of $D_z F(x_0)$ in the foldpoint computation (see later).

To determine which isotropy subgroup H satisfies the condition $\dim(Z_0 \cap X^H) = 1$, we usually search through the lattice of the isotropy subgroups of G starting from the root G. Note that when $\dim X^H = 1$ and $X^H \cap Z_0 \neq \{\phi\}$ then $q_H = 1$. An isotropy subgroup H is called maximal if it is not a proper subgroup of any other isotropy subgroup except G. If $\dim X^H = 1$ then H is a maximal isotropy subgroup. A major advantage of considering such maximal isotropy subgroups is that we can use (2.18) to determine all maximal isotropy subgroups H with $\dim X^H = 1$ and then use them as our first choice to test whether Theorem 3.8 is satisfied for H (see Sattinger [22] and Golubitsky *et al* [11]).

In view of Theorem 3.8, Golubitsky [9] conjectured that generically the solutions of symmetry breaking bifurcation problems all satisfy $\dim(X^H \cap Z_0) = 1$. It is now known that the conjecture is false. Counter–examples for higher dimensional representations of $SO(3)$ have been found by Chossat [2] and for higher dimensional representations of $O(3)$ by Lauterbach [13]. On the other hand, Field and Richardson [8] show that for finite groups generated by reflections the conjecture is true.

The above discussion about finding the symmetry breaking subgroup H of G is strongly representation dependent. If we change the representation we will need to rebuild the list of choices for H. Thus the question arises whether there exists any common list for different representations. A positive answer has been given by Rabier in [17] and [18]. More

specifically, it was shown that Lie groups possess subgroups of a special type, called intrinsic isotropy subgroups, which are characterized by a representation–independent property provided the representations are assumed to admit covariant linear isomorphisms with negative determinant. Such intrinsic isotropy subgroups play an important role in symmetry breaking bifurcations.

Let Γ be any Lie group and Γ' a subgroup of Γ. Then Γ' is an intrinsic isotropy subgroup if for any representation T of Γ in $GL(\mathbf{R}^n)$ with given $n \geq 1$, any T–covariant linear isomorphism $L \in GL(\mathbf{R}^n)$ satisfies

$$\operatorname{sgn}\ \det L_{\Gamma'} = \operatorname{sgn}\ \det L. \tag{3.47}$$

Suppose that the Lie group Γ is given along with one of its representations T in $GL(\mathbf{R}^n)$. We consider a C^1 mapping $g(= g(\mu, x)) : \mathbf{R} \times \mathbf{R}^n \mapsto \mathbf{R}^n$ such that $g(\mu, 0) = 0$ and assume that, for every $\mu \in \mathbf{R}, g(\mu, *)$ is Γ–covariant. Moreover, we assume that $\det D_x g(\mu, 0)$ changes sign as μ crosses 0. With these hypotheses, the following results hold.

Theorem 3.10 : ([17, Theorem 2.1]) *Let $\Gamma' \subset \Gamma$ be a (maximal) intrinsic isotropy subgroup. Then, there is a branch of nontrivial solutions of $g(\mu, x) = 0$ with Γ'–symmetry that bifurcates from* $(0,0)$.

For the existence of nontrivial intrinsic isotropy subgroups (the trivial case is $\Gamma' = \{I\}$), it was shown in [17] that a compact Lie group Γ has a nontrivial intrinsic isotropy subgroup if and only if it is not isomorphic to a direct product of copies of Z_2. If the elements of odd order in Γ form a subgroup Γ_{odd} (e.g. if Γ is supersolvable), then the subgroup (Γ^2) is an intrinsic isotropy subgroup of Γ, where (Γ^2) is the group generated by elements of the form $\gamma^2, \gamma \in \Gamma$. Surprisingly the converse is also true (see [18]).

Now we are ready to adapt the computational method for $(p, 1, 1)$ foldpoints to the reduced case of $(p_H, 1, 1)$ foldpoints. For this, we need the additional assumption

Assumption 3.11 : *There exist a point $x_0 \in M \cap X^G$ which is not a foldpoint with respect to Λ.*

Lemma 3.12 : *Under the Assumption 3.11 we have*

$$\dim Z^H = \dim Y^H, \text{ and } \dim\ker D_Z F_H(x) = \dim\ker D_Z F_H(x)^t. \tag{3.48}$$

Let $x^* \in X^H \cap M$ be a non–degenerate foldpoint of M with respect to Λ of type $(p, q, 1)$ where $q \geq 1$. Then we have rank $D_z F(x^*) = m - q$ and hence rank $D_z F(x) \geq m - q$ for all points x in some open subset U_0 of X containing x^*. Suppose further that $q_H = \dim\ker D_z F_H(x^*) = 1$ and set $U_0^H = U_0 \cap X^H$. Then we have rank $D_z F_H(x^*) = m_H - 1$ and hence rank $D_z F_H(x) \geq m_H - 1$ for all points x in some open subset V_0 of U_0^H containing x^*. In other words, for any foldpoint $x \in V_0 \cap M$ the first singularity index of F is at most q but the first singularity index of F_H equals 1.

Let L_H be an orthonormal matrix whose columns span Z^H, then from (3.48) the matrices $DF_H(x^*)L_H$ and $(DF_H(x^*)L_H)^t$ have one-dimensional nullspaces. Let $u_H^*, c_H^* \in \mathbf{R}^m$ be any given vectors such that

$$c_H^* \notin \operatorname{rge}\ DF_H(x^*)L_H, \quad u_H^* \notin \operatorname{rge}\ (DF_H(x^*)L_H)^t, \tag{3.49}$$

and for any $x \in V_0$ consider the matrix

$$A_H(x) = \begin{pmatrix} DF_H(x)L_H & c_H^* \\ (u_H^*)^t & 0 \end{pmatrix}. \tag{3.50}$$

Then there exists an open subset V_1 of V_0 containing x^* such that $A_H(x)$ is non–singular for all $x \in V_1$. Now the solutions of the two linear systems

$$A_H(x)\begin{pmatrix} u_H^0(x) \\ \mu_H(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, A_H(x)^t \begin{pmatrix} c_H^0(x) \\ \gamma_H(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, x \in V_1 \tag{3.51}$$

are uniquely determined for each $x \in V_1$. Applying Theorem 1 and Theorem 3 of [4] to the reduced $(p_H, 1, 1)$ foldpoint we have

Theorem 3.13 : *There exists an open neighborhood $V_2 \subset V_1$ of x^* such that the solutions of*

$$G_H(x) = \begin{pmatrix} F_H(x) \\ \gamma_H(x) \end{pmatrix}. \tag{3.52}$$

are exactly the foldpoints of M in $M_H \cap V_2$ with type $(p_H, 1, 1)$ and these solutions form a $(p_H - 1)$–dimensional submanifold of M_H.

Remark : Note that usually $p_H \leq p$, if $p_H = 2$, then the foldpoint submanifold with symmetry H is one–dimensional and we can apply PITCON (see [19]) to trace this submanifold. On the other hand, if $p_H \geq 3$, we can apply MATRIG (see [20]) to compute a simplicial approximation of an open subset of this sunmanfold.

We summarize the results in the form of the following algorithm.

Algorithm 3.14 : *Let x be an approximate foldpoint of type $(p, q, 1)$ of M with respect to Λ:*

(1) *Compute the singular value decomposition (SVD) of $D_z F(x)$*

$$B(x)^t D_z F(x) A(x) = \operatorname{diag}(\sigma_1(x), \ldots, \sigma_m(x)).$$

(2) *Let $a^i(x), i = 1, \ldots, m$ be the columns of the matrix $A(x)$ and compute the projection*

$$\Pi_{\ker D_z F(x)} = A_q A_q^t, \quad A_q = (a^{m-q+1}, \ldots, a^m).$$

(3) *For all possible symmetry breaking isotropy subgroups H*

3a) *compute the projection*

$$(P_Z)_H = \frac{1}{|H|} \sum_{h \in H} T_z(h);$$

3b) *and determine which isotropy subgroup H satisfies*

$$\dim(X^H \cap \ker D_Z F(x)) = \operatorname{tr}((P_Z)_H \Pi_{\ker D_Z F(x_0)}) = 1.$$

(4) *Compute* $a^i_H = (P_Z)_H a^i(x)$ *for* $i = m, \ldots, m-q+1$ *until* $a^{i_0}_H \neq 0$ *for some* i_0, *then take* $u^*_H = a^{i_0}_H$.
(5) *Compute the projection*

$$Q_H = \frac{1}{|H|} \sum_{h \in H} S(h).$$

(6) *Let* $b^j(x), j = 1, \ldots, m$ *be the columns of the matrix* $B(x)$ *and compute* $b^j_H = Q_H b^j(x)$ *for* $j = m, \ldots, m-q+1$ *until* $b^{j_0}_H \neq 0$ *for some* j_0, *then take* $c^*_H = b^{j_0}_H$.
(7) *Set* $x_0 = x$ *and for* $k = 0, 1, \ldots,$ *until convergence*
7a) *solve the systems*

$$A_H(x_k) \begin{pmatrix} u^0_H(x_k) \\ \mu_H(x_k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad A_H(x_k)^t \begin{pmatrix} c^0_H(x_k) \\ \gamma_H(x_k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0,$$

7b) *compute* $G_H(x_k)$ *and* $DG_H(x_k)$

$$G_H(x_k) = \begin{pmatrix} F_H(x_k) \\ \gamma_H(x_k) \end{pmatrix}$$

7c) *and compute the next iterate*

$$x_{k+1} = x_k - DG_H(x_k)^{-1} G_H(x_k)$$

4. Numerical example. A programming package BISYM has been written which implements the methods of Section 3. In order to indicate the performance of the package BISYM, we consider the four–box–Brusselator reaction problem (see [5]) in unfolded form

$$\begin{aligned} &\lambda A u - G(u,v) + \delta_1 a + \delta_2 b = 0 \\ &10\lambda A v - H(u,v) + 0.81533(\delta_1 a + \delta_2 b) = 0 \end{aligned} \qquad u, v \in \mathbf{R}^4 \tag{4.1}$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \tag{4.2}$$

and

$$\begin{aligned} &G(u,v) = (g(u_1,v_1), \ldots, g(u_4,v_4))^t, \quad g(s,t) = 2 - 6.9s + s^2 t \\ &H(u,v) = (h(u_1,v_1), \ldots, h(u_4,v_4))^t, \quad h(s,t) = 5.9 - s^2 t \\ &a = (0,1,0,-1)^t, \quad b = (1,0,-1,0)^t. \end{aligned} \tag{4.3}$$

With $x = (u, v, \lambda, \delta_1, \delta_2) \in \mathbf{R}^n$ and $n = 11, m = 8, p = 3$ this is a problem of the form considered in sections 2 and 3. It is evident that the equivariance relation (2.12) is satisfied with a group G that is isomorphic to the dihedral group D_4 :

$$G = \{I, R_1, R_2, R_3, S_1, S_2, S_3, S_4\}$$

$I, R_1, R_2, R_3, S_1, S_2, S_3, S_4$ are permutation matrices representing rotations and refections (see Figure 1). The induced group representation T_λ with respect to this unfolding is given by

$$T_\lambda(R_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} T_\lambda(R_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} T_\lambda(R_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.4}$$

and

$$\begin{aligned} T_\lambda(S_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} T_\lambda(S_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\ T_\lambda(S_3) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} T_\lambda(S_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{4.5}$$

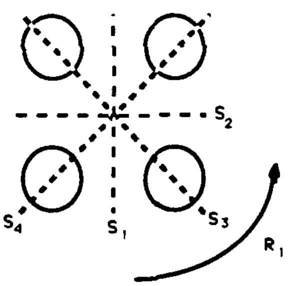


Figure 1

The subgroups of G are :

$$\begin{aligned} &G = \{I, R_1, R_2, R_3, S_1, S_2, S_3, S_4\}, \Sigma_R = \{I, R_1, R_2, R_3\}, \\ &\Sigma_M = \{I, R_2, S_1, S_2\}, \Sigma_D = \{I, R_2, S_3, S_4\}, \Sigma_\rho = \{I, R_2\}, \\ &\Sigma_1 = \{I, S_1\}, \Sigma_2 = \{I, S_2\}, \Sigma_3 = \{I, S_3\}, \Sigma_4 = \{I, S_4\}, \Sigma_0 = \{I\}. \end{aligned} \tag{4.5}$$

It can be shown that only $G, \Sigma_D, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ and Σ_0 are isotropy subgroups and $x_0 = (u^*, v^*, \lambda^*, 0, 0)$ with

$$e = (1, 1, 1, 1)^t, u^* = 2e, v^* = 2.95e, \lambda^* = 0.04536 \tag{4.7}$$

is a foldpoint of (4.1) with first and second singularity indices 2 and 1, respectively. By Theorem 3.8 and the formula (3.45), we can check that symmetries breaks from G to $\Sigma_i, i = 1, \ldots, 4$ respectively at x_0.

Using (4.4), (4.5) and (2.18) we obtain $\dim M_{\Sigma_i} = 2, i = 1, \ldots, 4$ (note $\dim M_{\Sigma_i} = \dim \Lambda^{\Sigma_i}$). Applying the subroutine PITCON of BISYM we can trace the foldlines in symmetry $\Sigma_i, i = 1, \ldots, 4$.

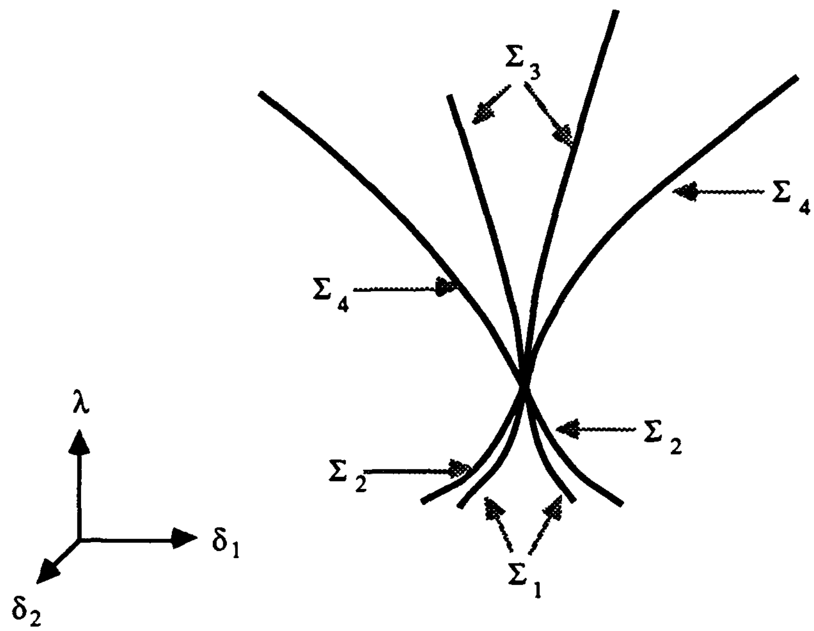
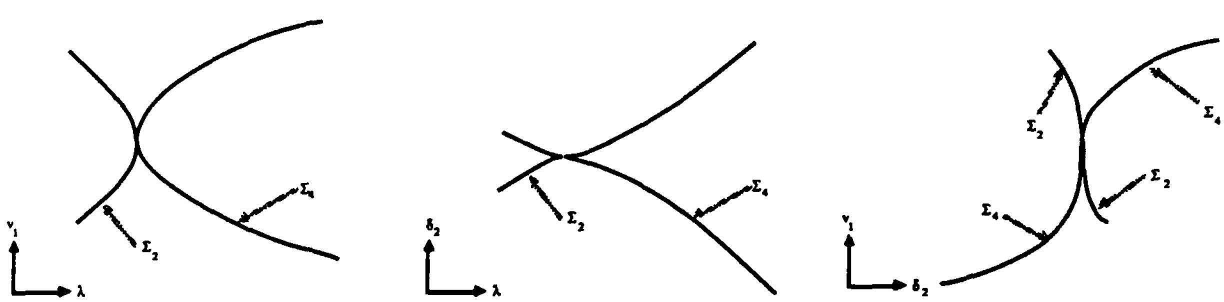


Figure 2

Figure 2 gives the projections of the foldlines; that is, the caustics of the symmetries $\Sigma_1, \Sigma_2, \Sigma_3$, and Σ_4 in the parameter space $(\delta_2, \delta_1, \lambda)$. Each caustic has the shape of a cusp. It may be interesting to note that the opening of the cusp of Σ_1, Σ_2 is opposite to that of Σ_3, Σ_4.



(a) (b) (c)

Figure 3

Figure 3 provides pictures of the projections of the foldlines of the symmetries Σ_2 and Σ_4 onto various 2–dimensional spaces, they indicate that the projections of foldlines of Σ_2 and Σ_4 onto the space of (v_1, λ, δ_2) behave as the cusp functions $v_1^3 + \lambda * v_1 + \delta_2$ and $v_1^3 - \lambda * v_1 + \delta_2$, respectively.

Applying the subroutine MATRIG of BISYM we can also compute the simplicial triangulation of the reduced manifold $M_{\Sigma_1}, \ldots, M_{\Sigma_4}$, respectively, in the coordinates (v_1, λ, δ_1) (see Figure 4).

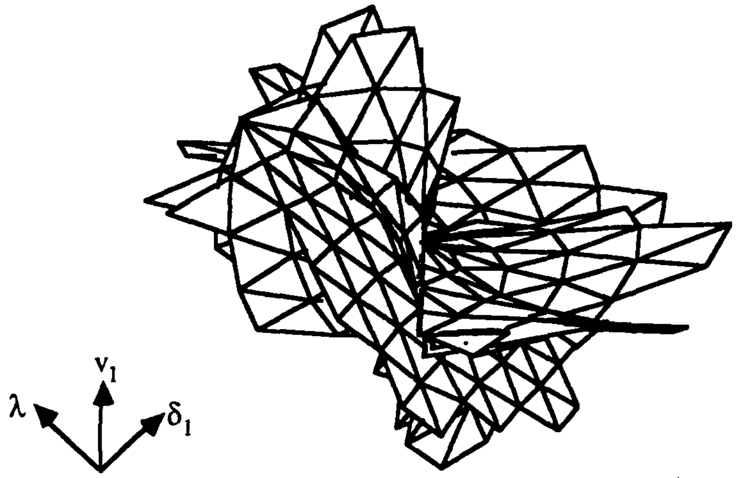


Figure 4

Figure 4 certainly indicates how complicated the solution behavior is in this example.

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13. ABSTRACT (*Maximum 200 words*)

This paper is about numerical methods for the determination of bifurcation points of certain steady state multi-parameter problems in the presence of symmetries. A principal tool is the fact that under general conditions the solution set forms a manifold in the space of all state and parameter variables. The reduced manifold with respect to some subsymmetry is introduced. Methods are presented for the local computation of the submanifold of bifurcation points of the same symmetry.

14. SUBJECT TERMS

solution manifold, symmetry groups, reduced form,foldpoint calculation

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